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L_p-error estimates for radial basis function interpolation on the sphere

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Abstract

In this paper we review the variational approach to radial basis function interpolation on the sphere and establish new L_p-error bounds, for $p \in [1, \infty]$. These bounds are given in terms of a measure of the density of the interpolation points, the dimension of the sphere and the smoothness of the underlying basis function.
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1. Introduction

Many applications lead to problems of interpolating values f_i of an unknown function f at scattered locations $x_i \in \mathbb{R}^d$ where $i = 1, \ldots, N$. One of the most promising ways of solving this problem is to employ *radial basis functions*. The most general radial basis function (RBF) interpolant takes the form

$$s(x) = \sum_{j=1}^{N} \lambda_j \phi(d(x, x_j)) + p(x), \tag{1.1}$$

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where $d(x,y) = ||x-y||$ is usually the Euclidean metric, p is a d-dimensional polynomial of suitable degree and $\phi : [0, \infty) \to \mathbb{R}$ is the RBF. The real coefficients $\{\lambda_j : j = 1, \dots, N\}$ and the polynomial p (if required) are chosen so that $s(x_i) = f_i$ $(1 \leqslant i \leqslant N)$.

Since the early 1990s progress in the interpolation theory of RBFs has been phenomenal. Encouraging theoretical findings regarding uniqueness, accuracy and stability have been discovered alongside ingenious numerical algorithms for practical implementation. For an excellent account of this material we recommend the textbook [2].

In this paper we are interested in examining how the RBF method can be used to solve the following spherical interpolation problem.

Problem. *Let* $\Xi = \{\xi_i\}_{i=1}^N$ *denote a set of distinct points located on the unit sphere* $S^{d-1} \subset \mathbb{R}^d$ *and let* $f(\xi_i)$ *denote the corresponding values of an unknown target function* $f : S^{d-1} \to \mathbb{R}$. *Find a continuous function* $s_f : S^{d-1} \to \mathbb{R}$ *which satisfies the following interpolation conditions*:

$$s_f(\xi_i) = f(\xi_i), \quad i = 1, \dots, N. \tag{1.2}$$

One way to attack this problem is to apply the RBF method directly. This is a perfectly valid approach, however it does not take into account the fact that the data points lie on a $(d-1)$-dimensional surface embedded in $\mathbb{R}^d$. A more illuminating approach is to specialise the method to the sphere. Specifically, we exchange the Euclidean metric for the geodesic metric defined by

$$g(\xi, \eta) = \cos^{-1}(\xi^T \eta), \quad \text{for } \xi, \eta \in S^{d-1}. \tag{1.3}$$

In addition we replace the RBF $\phi : [0, \infty) \to \mathbb{R}$ with a *zonal basis function* (ZBF) $\psi : [0, \pi] \to \mathbb{R}$, and, as a first step, we seek a basic interpolant of the form

$$s(\xi) = \sum_{j=1}^{N} \alpha_i \psi(g(\xi, \xi_j)), \quad \xi \in S^{d-1}. \tag{1.4}$$

In Section 2 of this paper we address the solvability of the spherical interpolation problem using ZBFs. Specifically, we briefly review the necessary functional analysis for the sphere and then, we introduce the class of strictly positive definite functions for the sphere. This class serves as a useful source of applicable basis functions for which a unique ZBF interpolant is guaranteed. Taking the RBF analogy one step further we dispense with the notion of polynomial reproduction and replace it with *spherical harmonic* polynomial reproduction, whereby we augment the basic ZBF interpolant by adding a spherical harmonic of a suitable degree. This leads us to introduce classes of conditionally strictly positive definite functions for the sphere, which again provide unique ZBF interpolants. To close Section 2 we demonstrate how each of the ZBF interpolants may also be viewed as the unique solution to a certain variational or minimal norm interpolation problem. We highlight the usefulness of this variational view by mentioning a pointwise error estimate which is a minor modification of the result found in [6].

Section 3 of the paper is devoted to convergence theory. If $\Xi = \{\xi_i\}_{i=1}^N$ denotes a set of scattered points on S^{d-1} then we measure the distribution of the Ξ by using the geodesic mesh norm

$$h = \sup_{\xi \in S^{d-1}} \min_{\xi_i \in \Xi} g(\xi, \xi_i). \tag{1.5}$$

In general, we aim to provide bounds of the form

$$||s_f - f||_{L_p(S^{d-1})} \leqslant \mathscr{B}(h)||f||, \quad p \in [1, \infty],$$

where f is measured in a suitable Sobolev-type norm and where $\mathscr{B}(h)$ is a function converging to zero as $h \to 0$, that is as the set Ξ becomes denser in S^{d-1}. Our approach is to revisit an old strategy originally used by Duchon [4] in his investigation of the accuracy of surface splines interpolants. In [14], Light and Wayne demonstrate how the Duchon strategy can be modified to provide error estimates for the more general case of RBF interpolation; see also [21]. In our companion paper [10] we have modified the Duchon strategy further so that it can work on the sphere. We rely on this framework together with the variational view of ZBF interpolation to provide new interpolation error estimates for sufficiently smooth target functions.

2. The zonal basis function method

2.1. Background

We begin by introducing *spherical harmonics*, which are the spherical analogue of classical polynomials. A good reference for this material is [17]. A spherical harmonic of order k on S^{d-1} is the restriction to S^{d-1} of a d-dimensional homogeneous harmonic polynomial of degree k. We let $\mathscr{H}_k^*(S^{d-1})$ denote the space of spherical harmonics of order k on S^{d-1}. This space has a useful intrinsic characterisation. If we let Δ_{d-1} denote the Laplace–Beltrami operator on S^{d-1} then the eigenvalues for the eigenvalue problem

$$(\Delta_{d-1} + \lambda)u = 0 \tag{2.1}$$

are $\lambda_k = k(k + d - 2)$ $k \geqslant 0$, and $\mathscr{H}_k^*(S^{d-1})$ is precisely the eigenspace of Δ_{d-1} corresponding to λ_k. The dimension $N_{k,d}$ of $\mathscr{H}_k^*(S^{d-1})$ is given by the multiplicity of λ_k in (2.1), specifically

$$N_{0,d} = 1, \quad \text{and} \quad N_{k,d} = \frac{2k + d - 2}{k}\binom{k + d - 3}{k - 1}, \quad k \geqslant 1.$$

Given an orthonormal basis $\{\mathscr{Y}_{k,l} : l = 1, \ldots, N_{k,d}\}$ for $\mathscr{H}_k^*(S^{d-1})$ the collection

$$\{\mathscr{Y}_{j,l} : l = 1, \ldots, N_{j,d} : j = 0, 1, \ldots, k\}$$

is an orthonormal basis for the space of spherical harmonics of order at most k, which we denote by $\mathscr{H}_k(S^{d-1})$. Furthermore, the collection

$$\{\mathscr{Y}_{j,l} : l = 1, \ldots, N_{j,d} : j \geqslant 0\}$$

forms an orthonormal basis for $L_2(S^{d-1})$. According to the celebrated Addition Theorem

$$P_{k,d}(\xi^T \eta) := \frac{\omega_{d-1}}{N_{k,d}} \sum_{l=1}^{N_{k,d}} \mathscr{Y}_{k,l}(\xi)\mathscr{Y}_{k,l}(\eta), \quad \xi, \eta \in S^{d-1}, \tag{2.2}$$

where $P_{k,d}$ is the d-dimensional Legendre polynomial of degree k, and where ω_{d-1} denotes the surface area of S^{d-1}.

Spherical harmonics can be used to give a "Fourier analysis" for the sphere. In particular, every function $f \in L_2(S^{d-1})$ has an associated Fourier series

$$f = \sum_{k=0}^{\infty} \sum_{l=1}^{N_{k,d}} \hat{f}_{k,l} \mathscr{Y}_{k,l}. \tag{2.3}$$

The Fourier coefficients are obtained by

$$\hat{f}_{k,l} = \int_{S^{d-1}} f(\xi)\mathscr{Y}_{k,l}(\xi)\, dS(\xi), \tag{2.4}$$

where dS represents a surface element of S^{d-1}. The square of the $L_2(S^{d-1})$-norm of f is given by

$$||f||^2_{L_2(S^{d-1})} = \int_{S^{d-1}} |f(\xi)|^2\, dS(\xi) = \sum_{k=0}^{\infty} \sum_{l=1}^{N_{k,d}} |\hat{f}_{k,l}|^2, \tag{2.5}$$

where the second equality is Parseval's identity.

For a real number $\beta \geqslant 0$ the Sobolev space $W_2^\beta(S^{d-1})$ of order β is defined as

$$\left\{ f \in L_2(S^{d-1}) : ||f||^2_{W_2^\beta(S^{d-1})} = \sum_{k=0}^{\infty} \sum_{l=1}^{N_{k,d}} (1 + \lambda_k)^\beta |\hat{f}_{k,l}|^2 < \infty \right\}. \tag{2.6}$$

The Sobolev embedding theorem holds true on the sphere. The theorem asserts that whenever $\beta > \frac{d-1}{2}$ then $W_2^\beta(S^{d-1})$ is continuously embedded in $C(S^{d-1})$.

The numbers $\lambda_k = k(k + d - 2)$ are the eigenvalues of the Laplace–Beltrami operator, which behave like k^2 for large k.

Remark 2.1. Let $\{\hat{c}_k\}_{k=0}^{\infty}$ denote a sequence of positive real numbers for which there exists positive constants c and C such that

$$\frac{c}{(1+k)^{2\beta}} \leqslant \hat{c}_k \leqslant \frac{C}{(1+k)^{2\beta}}, \quad k \geqslant 0, \ \beta > 0.$$

Then the space

$$\left\{ f \in L_2(S^{d-1}) : \|f\|_\beta^2 = \sum_{k=0}^{\infty} \sum_{l=1}^{N_{k,d}} \frac{|\hat{f}_{k,l}|^2}{\hat{c}_k} < \infty \right\}$$

is norm equivalent to the Sobolev space (2.6).

2.2. Interpolating with basis functions

In order to solve the spherical interpolation problem we propose the use of zonal basis functions and so we seek a basic interpolant with the form (1.4). Applying the interpolation conditions (1.2) we can deduce that the real coefficients $\{\alpha_i : i = 1, \ldots, N\}$ can be uniquely determined if and only if the interpolation matrix

$$A_{ij} = \psi(g(\xi_i, \xi_j)), \quad 1 \leqslant i, j \leqslant N \tag{2.7}$$

is non-singular,

Definition 2.2. A continuous function $\psi : [0, \pi] \to \mathbb{R}$ is said to be strictly positive definite on S^{d-1} ($\psi \in SPD(S^{d-1})$) if, for any set $\Xi = \{\xi_i\}_{i=1}^N$ of distinct points on S^{d-1}, the quadratic form

$$\alpha^T A \alpha = \sum_{j=1}^{N} \sum_{k=1}^{N} \alpha_j \alpha_k \psi(g(\xi_j, \xi_k)) \tag{2.8}$$

is positive on $\mathbb{R}^N \setminus \{0\}$.

If we choose $\psi \in SPD(S^{d-1})$, then the resulting interpolant (1.4) is unique since the corresponding interpolation matrix is, by definition, positive definite and hence non-singular.

Frequently one requires that an interpolant should reproduce the low order spherical harmonics. The ZBF interpolant s, given by (1.4), does not have this property and so it is often convenient to add to s a spherical harmonic of order k, which gives the form

$$s(\xi) = \sum_{j=1}^{N} \alpha_j \psi(g(\xi, \xi_j)) + \sum_{j=1}^{M} \beta_j \mathscr{Y}_j(\xi), \quad \xi \in S^{d-1}, \tag{2.9}$$

where $M = \dim \mathscr{H}_k(S^{d-1})$, and $\{\mathscr{Y}_1, \ldots, \mathscr{Y}_M\}$ is a basis for $\mathscr{H}_k(S^{d-1})$.

The interpolation conditions (1.2) now provide N linear equations in $N + M$ unknowns. In such cases it is usual to assume that $N \geqslant M$ and to impose M moment conditions on the $\{\alpha_i : i = 1, \ldots, N\}$ to take up the extra degrees of freedom.

Specifically, we use the equations

$$\sum_{j=1}^{N} \alpha_j \psi(g(\xi_i, \xi_j)) + \sum_{j=1}^{M} \beta_j \mathscr{Y}_j(\xi_i) = f(\xi_i), \quad 1 \leqslant i \leqslant N,$$
$$\sum_{j=1}^{N} \alpha_j \mathscr{Y}_i(\xi_j) = 0, \quad 1 \leqslant i \leqslant M. \tag{2.10}$$

Definition 2.3. For any set $\Xi = \{\xi_i\}_{i=1}^N$ of distinct data points on S^{d-1} we consider the following subspace:

$$W_{m-1} = \left\{ \alpha \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_i \mathscr{Y}(\xi_i) = 0 \text{ for all } \mathscr{Y} \in \mathscr{H}_{m-1}(S^{d-1}) \right\}. \tag{2.11}$$

A continuous function $\psi : [0, \pi] \to \mathbb{R}$ is said to be conditionally strictly positive definite of order $m \in \mathbb{N}$ on S^{d-1}, ($\psi \in CSPD_m(S^{d-1})$) if the quadratic form (2.8) is positive on $W_{m-1} \backslash \{0\}$.

Any function $\psi \in CSPD_m(S^{d-1})$ can be used to provide an augmented ZBF interpolant of the form (2.9) with $k = m - 1$. However, in order to guarantee the uniqueness of such a solution we require that the interpolation points satisfy the following geometric condition.

Spherical harmonic unisolvency. *Let $m \in \mathbb{N}$ and set $M = \dim \mathscr{H}_{m-1}(S^{d-1})$. A set of distinct points $\Xi = \{\xi_i\}_{i=1}^M$ is said to be $\mathscr{H}_{m-1}(S^{d-1})$-unisolvent if the only element of $\mathscr{H}_{m-1}(S^{d-1})$ to vanish at each ξ_i is the zero spherical harmonic.*

If $\psi \in CSPD_m(S^{d-1})$ and the interpolation points $\Xi = \{\xi_i\}_{i=1}^N$ contain an $\mathscr{H}_{m-1}(S^{d-1})$-unisolvent subset, then the interpolant of the form (2.9) is unique [13].

Using the work of Schoenberg [20], and extensions thereof [5], we can formulate the following theorem.

Theorem 2.4. *If $\psi \in CSPD_m(S^{d-1})$, then it has the following form*:

$$\psi(\theta) = \sum_{k=0}^{\infty} a_k P_{k,d}(\cos\theta), \tag{2.12}$$

where

$$a_k \geqslant 0 \quad \textit{for } k \geqslant m \quad \textit{and} \quad \sum_{k=0}^{\infty} a_k < \infty, \tag{2.13}$$

where $\{P_{k,d}\}$ denote the d-dimensional Legendre polynomials.

Remark 2.5. (i) In view of Theorem 2.4 we choose to consider each ZBF as a function of the inner product, $\xi^T \eta$, since $\cos(g(\xi, \eta)) = \xi^T \eta$.

(ii) Throughout this paper we shall take $\psi \in CSPD_0(S^{d-1})$ to mean $\psi \in SPD(S^{d-1})$. Further, if $\psi \in CSPD_m(S^{d-1})$ with $m > 0$, then we shall assume without loss that $a_k = 0$ for $0 \leqslant k \leqslant m-1$.

The complete characterisation of the class of functions of the form (2.12) satisfying (2.13) that are $CSPD_m(S^{d-1})$ has been investigated by several researchers see [15,18,19]. The most recent result is due to Chen et al. [3] who show that, for $d \geqslant 3$, a necessary and sufficient condition is that the set $\{k \in \mathbb{N}_0 \backslash \{0, 1, \ldots, m-1\} : a_k > 0\}$ must contain infinitely many odd and infinitely many even integers. The case of $d = 2$ remains an open problem and so we will only consider basis functions $\psi \in CSPD_m(S^{d-1})$ for which $a_k > 0$ for $k \geqslant m$.

2.3. A variational theory

For every $\psi \in CSPD_m(S^{d-1})$ we can associate a zonal kernel $\Psi(\xi, \eta) = \psi(\xi^T \eta)$. This, in turn, has a unique spherical Fourier expansion, given by

$$\Psi(\xi, \eta) = \sum_{k=m}^{\infty} \sum_{l=1}^{N_{k,d}} \hat{c}_k \mathscr{Y}_{k,l}(\xi) \mathscr{Y}_{k,l}(\eta), \tag{2.14}$$

where the $\hat{c}_k$ denote the spherical Fourier coefficients of Ψ. These are related, via the addition theorem, to the Legendre coefficients of ψ by $\hat{c}_k = a_k \omega_{d-1}/N_{kd}$. Furthermore, each sequence $\{\hat{c}_k\}_{k \geqslant m}$ possesses a certain decay rate as $k \to \infty$. In particular, we say that ψ has α-*Fourier decay* if there exists positive constants A_1, A_2 such that

$$A_1(1+k)^{-(d-1+\alpha)} \leqslant \hat{c}_k \leqslant A_2(1+k)^{-(d-1+\alpha)}, \quad \alpha > 0, \ \ k \geqslant m. \tag{2.15}$$

Definition 2.6. Let $\psi \in CSPD_m(S^{d-1})$ and let $\{\hat{c}_k\}_{k \geqslant m}$ denote the spherical Fourier coefficients of its associated zonal kernel (2.14). We define the native space of ψ to be

$$H_{\psi,m} := \left\{ f \in L_2(S^{d-1}) : |f|^2_{\psi,m} = \sum_{k=m}^{\infty} \sum_{l=1}^{N_{k,d}} \frac{|\hat{f}_{k,l}|^2}{\hat{c}_k} < \infty \right\}, \tag{2.16}$$

where $|\cdot|_{\psi,m}$ is a (semi-)norm induced via the (semi-)inner product

$$(f, g)_{\psi,m} = \sum_{k=m}^{\infty} \sum_{l=1}^{N_{k,d}} \frac{\hat{f}_{k,l} \hat{g}_{k,l}}{\hat{c}_k}. \tag{2.17}$$

If $m = 0$ then $|\cdot|_{\psi,0}$ is a norm which we rewrite as $||\cdot||_\psi$. Indeed, if ψ has α-Fourier decay, then Remark 2.1 allows us to deduce that $H_{\psi,0}$ is norm equivalent to the Sobolev space $W_2^\beta(S^{d-1})$ where $\beta = \frac{d-1+\alpha}{2}$.

If $m > 0$, then $|\cdot|_{\psi,m}$ is a semi-norm and has $\mathscr{H}_{m-1}(S^{d-1})$ as its null space. However, if we assume that $\{\xi_1, \ldots, \xi_M\}$ is a $\mathscr{H}_{m-1}(S^{d-1})$-unisolvent set, then we can consider a new bilinear form

$$\langle f, g \rangle_\psi = \sum_{i=1}^{M} f(\xi_i) g(\xi_i) + (f, g)_{\psi,m}, \quad f, g \in H_{\psi,m}. \tag{2.18}$$

With this in place we notice that $0 = \langle f, f \rangle_\psi$ if and only if $f \in \mathscr{H}_{m-1}(S^{d-1})$ and $\sum_{i=1}^{M} f(\xi_i)^2 = 0$, that is, if and only if $f = 0$. Thus, $\langle \cdot, \cdot \rangle_\psi$ is a genuine inner product for $H_{\psi,m}$. Furthermore, the space $H_{\psi,m}$ is complete with respect to the norm induced by $\langle \cdot, \cdot \rangle_\psi$ [13]. This allows us to consider the following new definition.

Definition 2.7. Let $m > 0$ and let $\psi \in CSPD_m(S^{d-1})$. We define the *normed* native Hilbert space of ψ by

$$H_\psi = \{ f \in L_2(S^{d-1}) : \|f\|_\psi < \infty \}, \tag{2.19}$$

where $\| \cdot \|_\psi$ is the norm induced by the inner product (2.18).

Before we discuss the importance of the native space we provide the following observation which will prove useful in Section 3, where we analyse the accuracy of the ZBF interpolants.

Observation 2.8. *If the basis function ψ has α-Fourier decay then we can use Remark* 2.1, *and the fact that all norms are equivalent on finite dimensional spaces*, *to deduce that H_ψ is norm equivalent to $W_2^\beta(S^{d-1})$, where $\beta = \frac{d-1+\alpha}{2}$. That is, there exists constants $0 < k_{\text{eq}} < K_{\text{eq}}$, such that*

$$k_{\text{eq}} \| \cdot \|_{W_2^\beta(S^{d-1})} \leqslant \| \cdot \|_\psi \leqslant K_{\text{eq}} \| \cdot \|_{W_2^\beta(S^{d-1})}. \tag{2.20}$$

In particular, *we can use the Sobolev embedding theorem to conclude that H_ψ is a Hilbert space of continuous functions.*

The importance of the native space of a basis function $\psi \in SCPD_m(S^{d-1})$ is well illustrated by Levesley et al. [13], where it is shown that, given any $f \in H_\psi$, the solution to the following variational problem:

$$\text{minimise } \{ \|s\|_\psi : s \in H_\psi \text{ and } s(\xi_i) = f(\xi_i) \ 1 \leqslant i \leqslant N \}, \tag{2.21}$$

is precisely the unique ψ-based ZBF interpolant. This variational problem is precisely the same as finding the optimal interpolant in a Hilbert space, such problems are well understood and were studied in the late 1950s by Golomb and Weinberger [7]. The real power of the variational approach lies in the fact that the original Hilbert space techniques from [7] can be applied to provide useful pointwise error bounds. Specifically, for a given $f \in H_\psi$, the error of its ψ-based ZBF interpolant s_f can be bounded by an estimate of the form

$$|s_f(\xi) - f(\xi)| \leqslant P_\psi(\xi) \cdot \|s_f - f\|_\psi, \quad \xi \in S^{d-1}. \tag{2.22}$$

The factor $P_\psi(\xi)$ is called the *power function* of $\psi \in CSPD_m(S^{d-1})$ and has the following explicit form:

$$P_\psi(\xi) = \left(\sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_i \gamma_j \psi(\xi_i^T \xi_j) - 2 \sum_{i=1}^{N} \gamma_i \psi(\xi^T \xi_i) + \psi(1) \right)^{1/2},$$

where the coefficients $\{\gamma_i \in \mathbb{R} : i = 1, \dots, N\}$ are chosen to satisfy

$$\mathscr{Y}(\xi) = \sum_{i=1}^{N} \gamma_i \mathscr{Y}(\xi_i), \quad \text{for all } \mathscr{Y} \in \mathscr{H}_{J-1}(S^{d-1}), \tag{2.23}$$

where $J \geqslant m$ is a fixed integer. The *optimal* power function $P_\psi^*(\xi)$ is determined by minimising the quadratic expression for $P_\psi(\xi)$ over all coefficients $\{\gamma_i\}_{i=1}^N$ which satisfy (2.23) [22]. Stated in this way, it is clear that a close investigation of P_ψ, and especially the choice of the γ_i, ought to provide an insight into the accuracy of the ZBF interpolation method. Indeed this strategy is employed, in quite different ways, by Jetter et al. [11] and also by von Golitschek and Light [6] to provide error bounds of the form

$$|s_f(\xi) - f(\xi)| \leqslant \mathscr{B}(h) \cdot ||s_f - f||_\psi, \quad \xi \in S^{d-1}, \tag{2.24}$$

where h is the geodesic mesh norm defined by (1.5) and $\mathscr{B}(h) \to 0$ as $h \to 0$.

We remark that the error bound (2.22) may be viewed as a specific instance of the following more general result.

Proposition 2.9. *Let $\psi \in CSPD_m(S^{d-1})$ and let $\Xi = \{\xi_i\}_{i=1}^N$ denote a set of distinct points on S^{d-1}. Consider the subspace*

$$Z_\psi = \{f \in H_\psi : f(\xi_i) = 0 \quad i = 1, \dots, N\},$$

then

$$|f(\xi)| \leqslant P_\psi(\xi) \cdot ||f||_\psi, \quad \text{for all } f \in Z_\psi \quad \text{and} \quad \xi \in S^{d-1}. \tag{2.25}$$

So far in this paper we have alluded to the use of the geometric mesh-norm h (1.5) to measure the relative density of a set of data points $\Xi = \{\xi_i\}_{i=1}^N$ in S^{d-1}. Geometrically speaking, h represents the radius of the largest spherical cap (open geodesic ball) which can be placed on S^{d-1} without covering any $\xi_i \in \Xi$. In [6], von Golitschek and Light use the height h_d of the maximal spherical cap as an alternative mesh-norm. That is, they define h_d to be the smallest number such that

$$\inf_{\eta \in S^{d-1}} \max\{\eta^T \xi_i : \xi_i \in \Xi\} > 1 - h_d \tag{2.26}$$

is satisfied. We shall call h_d the "dot product" mesh norm of Ξ. Using some elementary trigonometry we can show that $h_d = 2\sin^2(h/2)$. Furthermore, if $h \in (0, 2\pi/3)$ then we can apply the small angle result for $\sin(h/2)$ to give

$$\frac{h^2}{8} \leqslant h_d \leqslant \frac{h^2}{2} \tag{2.27}$$

that is, h_d is equivalent to h^2. The idea of using the dot product as an alternative measure of distance will prove to be a useful one.

Definition 2.10. For every $\xi \in S^{d-1}$ we define an associated a dot-product distance function

$$d_\xi : S^{d-1} \to [-1, 1], \text{ given by } d_\xi(\eta) = \xi^T \eta.$$

Furthermore, we can define a dot product neighbourhood of ξ by

$$N(\xi, r_d) = \{\eta \in S^{d-1} : d_\xi(\eta) > 1 - r_d\}, \quad \text{where } r_d \in (0, 1). \tag{2.28}$$

The following crucial result is quoted from [6].

Lemma 2.11. *Let J be a fixed positive integer and let $\Xi = \{\xi_1, \ldots, \xi_N\}$ denote a set of N distinct data points in S^{d-1} with dot product mesh-norm h_d. There is a number $h_0 \in (0, 1)$ such that if $h_d < h_0$, and $\xi \in S^{d-1}$, then there exist coefficients $\{\gamma_i\}_{i=1}^N$ such that*

1. $\mathscr{Y}(\xi) = \sum_{i=1}^N \gamma_i \mathscr{Y}(\xi_i)$, *for all* $\mathscr{Y} \in \mathscr{H}_{J-1}(S^{d-1})$,
2. *there exists a constant K_1 (independent of ξ and h_d) such that if $\xi_i \notin N(\xi, K_1 h_d)$, then $\gamma_i = 0$, and*
3. *there exists a constant K_2 (independent of ξ and h_d) such that $\sum_{i=1}^N |\gamma_i| \leqslant K_2$.*

With this preparation the following result can be established.

Theorem 2.12. *Let $\psi \in CSPD_m(S^{d-1})$ have α-Fourier decay and let $\Xi = \{\xi_i\}_{i=1}^N$ denote a set distinct points on S^{d-1}. Set*

$$J = \max\left\{m, \frac{\lceil \alpha + 1 \rceil}{2}\right\}, \tag{2.29}$$

where $\lceil x \rceil$ denotes the smallest integer $\geqslant x$, and assume that the mesh-norm h (1.5) of Ξ satisfies

$$\frac{1}{K+1} \leqslant h < \frac{1}{K}, \tag{2.30}$$

where $K > J$ is a positive integer. Let $f \in H_\psi$ and s_f denote its unique ZBF interpolant. Then, for any $\xi \in S^{d-1}$, we have

$$|f(\xi) - s_f(\xi)| \leqslant \mathscr{C} \cdot h^{\alpha/2} \cdot ||f - s_f||_\psi, \tag{2.31}$$

where $\mathscr{C}$ is a positive constant independent of h.

Proof. For a full proof of this result see [16, Theorem 2]. For a brief sketch, we note that choice of integer J allows us to evoke Lemma 2.11 to provide, for any $\xi \in S^{d-1}$, a neighbourhood $N(\xi, K_1 h_d)$ and a set of local coefficients $\{\gamma_i\}_{i \in I_{\text{loc}}}$, where $I_{\text{loc}} := \{i : \xi_i \in \Xi \cap N(\xi, K_1 h_d)\}$, which satisfy condition (2.23). Furthermore, these coefficients can be used to define a local power function, $P_{\psi,\mathrm{loc}}$ say, which by (2.22), provides the pointwise error bound

$$|s_f(\xi) - f(\xi)| \leqslant P_{\psi,\mathrm{loc}}(\xi) \cdot ||s_f - f||_\psi, \quad \xi \in S^{d-1}. \tag{2.32}$$

It then remains to show that $P_{\psi,\mathrm{loc}}$ can be bounded above by a constant multiplied by $h^{\alpha/2}$. □

We close this section by providing two important properties of the ZBF interpolant, both of which can be inferred from the theory of optimal interpolation in a Hilbert space [7].

Lemma 2.13. *Let $\psi \in CSPD_m(S^{d-1})$. For a given $f \in H_\psi$ let s_f denote its unique ψ-based ZBF interpolant, then we have*

$$\text{(i)}\ ||f - s_f||_\psi^2 = \langle f, f - s_f \rangle_\psi, \quad \text{(ii)}\ ||f - s_f||_\psi \leqslant ||f||_\psi.$$

3. Global error estimates

In this section we generalise techniques dating back to Duchon [4], from his study of the accuracy of interpolation using surface splines in Euclidean space. The requirements for a Duchon framework for the sphere are as follows:

(i) A suitable quasi-uniform mesh of data points for the sphere.
(ii) A suitable Sobolev extension operator for the sphere.
(iii) A spherical version of Duchon's inequality.

The technical effort required to establish these items is quite considerable. In view of this, we shall simply state the key results and refer the reader to our accompanying paper [10] for full details.

3.1. The key results

3.1.1. A quasi-uniform mesh for the sphere

Lemma 3.1. *Let $d \geqslant 2$, be an integer and set*

$$M = 2\sqrt{d-1} \quad \textit{and} \quad \delta_d = \frac{1}{4d^{3/2}}.$$

Let M_1 be an arbitrary positive number, $\theta \in (0, \pi/3)$ and set

$$h_0 := \frac{\theta}{M + M_1 + \delta_d}. \tag{3.1}$$

Then, for any $h \in (0, h_0)$, there exists a set of points $Z_h \subset S^{d-1}$ such that

$$S^{d-1} = \bigcup_{z \in Z_h} G(z, Mh).$$

Let F_A denote the characteristic function of a set $A \subset S^{d-1}$. There exists a positive integer Q independent of h such that

$$\sum_{z \in Z_h} F_{G(z,\bar{M}h)} \leqslant Q, \quad \text{where } \bar{M} = M + M_1. \tag{3.2}$$

Further, the cardinality of Z_h is bounded above by $C_Q h^{-(d-1)}$, where C_Q is independent of h.

Proof. Lemma 3.1 [10]. □

3.1.2. A Sobolev extension theorem for the sphere

Theorem 3.2. *Let $z \in S^{d-1}$ and $\Xi = \{\xi_i\}_{i=1}^N$ denote a set of distinct points on S^{d-1}. Let $\beta \in [k, k+1]$, where $k > \frac{d-1}{2}$ is a positive integer. There exists positive numbers $\mathcal{R}_0$ and $C_{\mathcal{A}}$ such that if we let $M_1 > \max\{\mathcal{R}_0 - 2\sqrt{d-1}, 0\}$ be a fixed positive number and let*

$$h_0 = C_{\mathcal{A}}/(3\bar{M}) \quad \text{where } \bar{M} = 2\sqrt{d-1} + M_1, \tag{3.3}$$

then, assuming that Ξ has mesh norm $h \in (0, h_0)$, there exists an extension operator $E_{G(z,\bar{M}h)} : W_2^\beta(G(z, \bar{M}h)) \to W_2^\beta(S^{d-1})$ satisfying

(1) $(E_{G(z,\bar{M}h)} f)|_{G(z,\bar{M}h)} = f$, *for all* $f \in W_2^\beta(G(z, \bar{M}h))$,
(2) *there exists a positive constant $\mathcal{K}$, independent of h and z such that*

$$\|E_{G(z,\bar{M}h)} f\|_{W_2^\beta(S^{d-1})} \leqslant \mathcal{K} \cdot \|f\|_{W_2^\beta(G(z,\bar{M}h))},$$

for all $f \in W_2^\beta(G(z, \bar{M}h))$ such that $f(\xi) = 0$ for $\xi \in \Xi \cap G(z, \bar{M}h)$.

Proof. Theorem 5.19 [10]. □

3.1.3. A spherical version of Duchon's inequality

Theorem 3.3. *Let $\beta > 0$ and let M_1 be any positive number. Set h_0 to be as in* (3.3), *let $h \in (0, h_0)$ and let Z_h denote the corresponding quasi-uniform mesh for S^{d-1} from Lemma* 3.1. *Then, for any $f \in W_2^\beta(S^{d-1})$, we have*

$$\sum_{z \in Z_h} \|f\|^2_{W_2^\beta(G(z,\bar{M}h))} \leqslant Q \|f\|^2_{W_2^\beta(S^{d-1})}, \tag{3.4}$$

where Q is the constant (*independent of h*) *from Lemma* 3.1.

Proof. Theorem 3.2 [10]. □

3.2. Global error bounds

The three technical results stated in the previous subsection, i.e., the Duchon framework for the sphere, can be viewed as the key ingredients of a recipe for providing error bounds. In this section we demonstrate exactly how this framework can be used to provide new L_p-error bounds for ZBF interpolation. We begin by stating our aim.

Assume that $\psi \in CSPD_m(S^{d-1})$ has α-Fourier decay. Then given any target function $f \in H_\psi$ and a set $\Xi = \{\xi_i\}_{i=1}^N$ of distinct interpolation nodes with geodesic mesh norm h (1.5), our aim is to examine the accuracy of the ψ-based interpolant s_f to f over Ξ. Specifically, we aim to establish bounds of the following form:

$$||f - s_f||_{L_p(S^{d-1})} \leqslant \mathscr{C} h^{\lambda_p} ||f - s_f||_\psi,$$

where the constant $\mathscr{C}$ is independent of f and h, and where the number $\lambda_p > 0$ is the called the L_p-convergence order.

Our first task is to use the covering of S^{d-1} (see Lemma 3.1) to write

$$\begin{aligned} ||f - s_f||^p_{L_p(S^{d-1})} &= \int_{S^{d-1}} |(f - s_f)(\xi)|^p \, dS(\xi) \\ &\leqslant \sum_{z \in Z_h} \int_{G(z,Mh)} |(f - s_f)(\xi)|^p \, dS(\xi), \quad \text{for } M = 2\sqrt{d-1}. \end{aligned}$$

This step gives us the advantage that we can consider the error locally. In particular, the function $f - s_f$ is continuous on $\overline{G(z, Mh)}$ and, as this is a compact subset of S^{d-1}, there exists a point $\xi_z \in \overline{G(z, Mh)}$ at which $f - s_f$ attains its maximum. This observation allows us to write

$$\begin{aligned} ||f - s_f||^p_{L_p(S^{d-1})} &\leqslant \sum_{z \in Z_h} |(f - s_f)(\xi_z)|^p \int_{G(z,Mh)} dS(\xi) \\ &\leqslant C_d h^{d-1} \sum_{z \in Z_h} |(f - s_f)(\xi_z)|^p, \end{aligned} \tag{3.5}$$

where C_d is a constant depending only on d which satisfies

$$\text{Area}(G(z, Mh)) \leqslant C_d h^{d-1}. \tag{3.6}$$

We know, from the variational theory, that $f - s_f \in H_\psi$. Furthermore, since ψ has α-Fourier decay then its native space H_ψ is norm equivalent to the Sobolev space $W_2^\beta(S^{d-1})$, where $2\beta = \alpha + d - 1$ (see Remark 2.1). Now, rather than consider $f - s_f$, we choose instead to consider the restriction $f - s_f|_{G(z,\bar{M}h)}$ where $\bar{M} = 2\sqrt{d-1} + M_1$, for some $M_1 > 0$ whose value, or more precisely, whose range of values, is not yet determined.

In choosing a suitable value for M_1, and hence $\bar{M}$, we must take into account the following conditions:

(a) In order to employ Theorem 2.12 to provide pointwise error estimates, we require that each $G(z, \bar{M}h)$ must contain the dot product neighbourhood $N(\xi_z, K_1 h_d)$.

(b) In order to apply the Sobolev extension operator to $f - s_f|_{G(z,\bar{M}h)} \in W_2^{\beta}(G(z,\bar{M}h))$, we require that $\bar{M}h \in (\mathscr{R}_0 h, C_{\mathscr{A}}/3)$.

If we let $\mathscr{R}_0$ and $C_{\mathscr{A}}$ denote the constants from Theorem 3.2 corresponding to $\beta = \frac{\alpha+d-1}{2}$, then the condition $M_1 > \max\{\mathscr{R}_0 - 2\sqrt{d-1}, 0\}$ together with the assumption that the geodesic mesh norm of Ξ should satisfy

$$0 < h < C_{\mathscr{A}}/(3\bar{M}) \tag{3.7}$$

are sufficient to guarantee that condition (b) is satisfied (see Theorem 3.2).

We now turn to condition (a). Let K_1 denote the neighbourhood constant from Theorem 2.12. For any ξ_z, the neighbourhood $N(\xi_z, K_1 h_d)$ can also be viewed, in more familiar terms, as an open geodesic ball $G(\xi_z, \theta)$, where θ satisfies $\sin^2(\theta/2) = K_1 h_d/2$.

If we assume that the dot product mesh norm (2.26) satisfies

$$h_d < h_0^{(\mathrm{dot})} = 3/(2K_1), \tag{3.8}$$

then we have that $\theta \in (0, 2\pi/3)$, thus we can apply the small angle result for $\sin(\theta/2)$, followed by the mesh-norm equivalence relation (2.27) to deduce that

$$\frac{\theta}{2\sqrt{2}} \leqslant \sqrt{K_1 h_d} \leqslant \sqrt{K_1} \cdot \frac{h}{\sqrt{2}}.$$

In particular, if M_1, is chosen according to

$$M_1 > \max\{\mathscr{R}_0 - 2\sqrt{d-1}, 2\sqrt{K_1}\}, \tag{3.9}$$

then this shows that

$$\begin{aligned} N(\xi_z, K_1 h_d) = G(\xi_z, \theta) &\subset G(\xi_z, 2\sqrt{K_1} h) \\ &\subset G(z, (M + 2\sqrt{K_1})h) \\ &\subset G(z, (M + M_1)h) = G(z, \bar{M}h) \\ &\subset G(z, C_{\mathscr{A}}/3), \quad \text{(see Fig.1)}. \end{aligned}$$

And so both conditions (a) and (b) are simultaneously satisfied.

Let $v_z = f - s_f|_{G(z,\bar{M}h)}$ then, using the Sobolev extension operator, we have

E1. $E_{G(z,\bar{M}h)} v_z \in W_2^{\beta}(S^{d-1})$.

E2. $E_{G(z,\bar{M}h)} v_z(\xi) = 0$ for all $\xi \in \Xi \cap G(z, \bar{M}h)$.

E3. Using part 2 of Theorem 3.2, there exists a constant $\mathscr{K}$, independent of h and z such that $||E_{G(z,\bar{M}h)} v_z||_{W_2^{\beta}(S^{d-1})} \leqslant \mathscr{K} \cdot ||v_z||_{W_2^{\beta}(G(z,\bar{M}h))}$.

Since condition (a) is satisfied, we have that

$$\Xi_{\xi_z} = \Xi \cap N(\xi_z, K_1 h_d) \subset \Xi \cap G(z, \bar{M}h) = \Xi_z.$$

Thus, the optimal power function of ψ, based upon Ξ_z and evaluated at the point ξ_z, can be bounded above by the local power function $P_{\psi,\mathrm{loc}}(\xi_z)$. Moreover, if we let J

denote the integer from Theorem 2.12 and we assume that the geodesic mesh norm of Ξ satisfies

$$h \in (0, h_0^{(\text{geod})}) \quad \text{where } h_0^{(\text{geod})} = \min\left(\frac{C_{\mathscr{A}}}{3\bar{M}}, \frac{1}{K}\right), \tag{3.10}$$

where $K > J$ is a positive integer, then we can subsequently use the local error bound (2.31) together with (2.20) and E3, respectively, to yield

$$\begin{aligned} |(f - s_f)(\xi_z)| &= |E_{G(z,\bar{M}h)} v_z(\xi_z)| \leqslant P_{\psi,\text{loc}}(\xi_z) ||E_{G(z,\bar{M}h)} v_z||_\psi \\ &\leqslant P_{\psi,\text{loc}}(\xi_z) K_{\text{eq}} ||E_{G(z,\bar{M}h)} v_z||_{W_2^\beta(S^{d-1})} \\ &\leqslant \mathscr{C} K_{\text{eq}} \cdot h^{\frac{\alpha}{2}} ||E_{G(z,\bar{M}h)} v_z||_{W_2^\beta(S^{d-1})} \\ &\leqslant \mathscr{K}\mathscr{C} K_{\text{eq}} \cdot h^{\frac{\alpha}{2}} ||v_z||_{W_2^\beta(G(z,\bar{M}h))}. \end{aligned}$$

Substituting this into (3.5) gives

$$||f - s_f||^p_{L_p(S^{d-1})} \leqslant C_d (\mathscr{K}\mathscr{C} K_{\text{eq}})^p \cdot h^{(\frac{\alpha p}{2} + d - 1)} \sum_{z \in Z_h} ||f - s_f|_{G(z,\bar{M}h)}||^p_{W_2^\beta(G(z,\bar{M}h))}.$$

For $p \geqslant 2$ we use Jensen's inequality $\sum_{i=1}^N a_i^p \leqslant (\sum_{i=1}^N a_i^2)^{p/2}$ [1], followed by Theorem 3.3, and (2.20) to give

$$\begin{aligned} &||f - s_f||^p_{L_p(S^{d-1})} \\ &\leqslant C_d (\mathscr{K}\mathscr{C} K_{\text{eq}})^p \cdot h^{(\frac{\alpha p}{2} + d - 1)} \cdot \left(\sum_{z \in Z_h} ||(f - s_f)|_{G(z,\bar{M}h)}||^2_{W_2^\beta(G(z,\bar{M}h))} \right)^{p/2} \end{aligned}$$

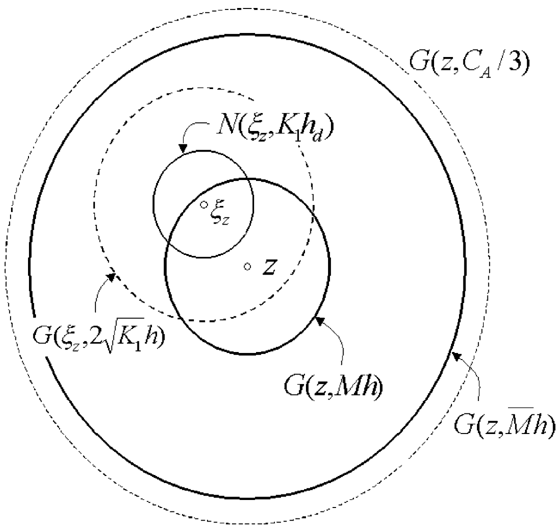

Fig. 1. Illustration of the nesting of the key neighbourhoods.

$$\leqslant C_d(\mathcal{K}\mathcal{C}K_{\text{eq}}\sqrt{Q})^p \cdot h^{(\frac{\alpha p}{2}+d-1)} \cdot (||f-s_f||^2_{W_2^\beta(S^{d-1})})^{p/2}$$
$$\leqslant C_d(\mathcal{K}\mathcal{C}K_{\text{eq}}\sqrt{Q}k_{\text{eq}}^{-1})^p \cdot h^{(\frac{\alpha p}{2}+d-1)} \cdot ||f-s_f||^p_\psi.$$

Finally, taking the pth root gives

$$||f-s_f||_{L_p(S^{d-1})} \leqslant C \cdot h^{\frac{\alpha}{2}+\frac{d-1}{p}} ||f-s_f||_\psi,$$

where $C = C_d^{1/p}(\mathcal{K}\mathcal{C}K_{\text{eq}}\sqrt{Q}k_{\text{eq}}^{-1})$ is independent of f and h.

For $p \in [1,2)$ we execute the same arguments as above, however we replace Jensen's inequality with $\sum_{i=1}^N a_i^p \leqslant N^{1-\frac{p}{2}}(\sum_{i=1}^N a_i^2)^{p/2}$ [1]. Further, we use the fact that the cardinality of Z_h is bounded by $C_Q h^{-(d-1)}$, see Lemma 3.1, to deduce that

$$||f-s_f||^p_{L_p(S^{d-1})}$$
$$\leqslant C_d(\mathcal{K}\mathcal{C}K_{\text{eq}})^p \cdot h^{(\frac{\alpha p}{2}+d-1)} \cdot \left(\sum_{z \in Z_h} ||(f-s_f)|_{G(z,\bar{M}h)}||^2_{W_2^\beta(G(z,\bar{M}h))}\right)^{p/2}$$
$$\leqslant C_d(\mathcal{K}\mathcal{C}K_{\text{eq}}\sqrt{Q})^p C_Q \cdot h^{p(\frac{\alpha+d-1}{2})} \cdot (||f-s_f||^2_{W_2^\beta(S^{d-1})})^{p/2}$$
$$= C_d(\mathcal{K}\mathcal{C}K_{\text{eq}}\sqrt{Q}k_{\text{eq}}^{-1})^p C_Q \cdot h^{p(\frac{\alpha+d-1}{2})} \cdot ||f-s_f||^p_\psi.$$

Finally, taking the pth root provides

$$||f-s_f||_{L_p(S^{d-1})} \leqslant C \cdot h^{\frac{\alpha+d-1}{2}} ||f-s_f||_\psi,$$

where $C = (C_d C_Q)^{1/p}(\mathcal{K}\mathcal{C}K_{\text{eq}}\sqrt{Q}k_{\text{eq}}^{-1})$ is independent of f and h.

In summary we have proved the following theorem.

Theorem 3.4. *Assume that $\psi \in CSPD_m(S^{d-1})$ has α-Fourier decay. Let Ξ denote a set of distinct data points on S^{d-1} with geodesic mesh-norm h* (1.5). *There exists a positive number h_0 such that, if $h \in (0,h_0)$ then the ψ-based interpolant s_f to any target function $f \in H_\psi$ satisfies*

$$||f-s_f||_{L_p(S^{d-1})} \leqslant C \cdot h^{\frac{\alpha}{2}+\frac{d-1}{p}} ||f-s_f||_\psi, \quad \text{for } p \in [2,\infty) \tag{3.11}$$

and

$$||f-s_f||_{L_p(S^{d-1})} \leqslant C \cdot h^{\frac{\alpha}{2}+\frac{d-1}{2}} ||f-s_f||_\psi, \quad \text{for } p \in [1,2), \tag{3.12}$$

where the generic constant C is independent of f and h.

Proof. Let M_1 be chosen according to (3.9) and define

$$h_0 = \min\left(h_0^{(\text{geod})}, \sqrt{2h_0^{(\text{dot})}}\right),$$

where $h_0^{(\mathrm{geod})}$ and $h_0^{(\mathrm{dot})}$ are defined by (3.10) and (3.8), respectively. If $h \in (0, h_0)$, then the two density conditions (3.10) and (3.8) are satisfied. Thus, the arguments set out in the analysis leading up to the theorem can be employed to provide the desired results. □

Remark 3.5. The constants appearing in the error bounds (3.11) and (3.12) depend on the value of p. For $p \in [2, \infty]$ the dependence is due to the factor $C_d^{1/p}$ where C_d is given by (3.6). For $p \in [1, 2)$ the dependence is due to the factor $(C_d C_Q)^{1/p}$ where C_Q is taken from Lemma 3.1, In both cases we note that the constants do not grow excessively large as p varies.

3.3. Improved global error bounds

At first glance it is tempting to "tidy up" the error results from Theorem 3.4 by employing the optimality bound $\|f - s_f\|_\psi \leqslant \|f\|_\psi$, from Lemma 2.13(ii). This is a perfectly valid procedure, however we will show that an improved bound is available, provided that f belongs to a certain subspace of H_ψ, which we shall denote as $H_{\psi*\psi}$. Once this improved bound is established we will use it to improve the L_p-convergence order in (3.11) for target functions $f \in H_{\psi*\psi}$.

Definition 3.6. Let $\psi \in CSPD_m(S^{d-1})$ have α-Fourier decay and let Ψ denote its corresponding zonal kernel. We define the convolution kernel of Ψ by

$$(\Psi * \Psi)(\xi, \eta) := \int_{S^{d-1}} \Psi(\xi, v)\Psi(v, \eta)\, d\omega_{d-1}(v), \quad \xi, \eta \in S^{d-1}.$$

It is more revealing to work in terms of Fourier expansions since we have

$$\Psi(\xi, \eta) = \sum_{k=m}^{\infty} \sum_{l=1}^{N_{k,d}} \hat{c}_k \mathscr{Y}_{k,l}(\xi)\mathscr{Y}_{k,l}(\eta) \Rightarrow (\Psi * \Psi)(\xi, \eta) = \sum_{k=m}^{\infty} \sum_{l=1}^{N_{k,d}} \hat{c}_k^2 \mathscr{Y}_{k,l}(\xi)\mathscr{Y}_{k,l}(\eta).$$

This observation allows us to define a convolution native space by

$$H_{\psi*\psi,m} = \left\{ f \in L_2(S^{d-1}) : |f|_{\psi*\psi,m} = \left(\sum_{k=m}^{\infty} \sum_{l=1}^{N_{k,d}} \frac{|\hat{f}_{k,l}|^2}{\hat{c}_k^2} \right)^{1/2} < \infty \right\}.$$

The observations made in Section 2, regarding native spaces, also apply to convolution native spaces. In particular, we can define a normed native space $(H_{\psi*\psi}, \|\cdot\|_{\psi*\psi})$ and conclude that

$$(H_{\psi*\psi}, \|\cdot\|_{\psi*\psi}) \cong W_2^{2\beta}(S^{d-1}) \subset W_2^{\beta}(S^{d-1}) \cong (H_\psi, \|\cdot\|_\psi), \tag{3.13}$$

where $\beta = \frac{\alpha+d-1}{2}$ and where $\cong$ denotes norm equivalence.

Lemma 3.7. *For a given $f \in H_{\psi*\psi}$, let s_f denote its unique ψ-based ZBF interpolant. Then*

$$||f - s_f||_\psi^2 \leqslant ||f||_{\psi*\psi} \cdot ||f - s_f||_{L_2(S^{d-1})}. \tag{3.14}$$

Proof. Using Lemma 2.13(i), the definition of $\langle \cdot, \cdot \rangle_\psi$ and an application of the Cauchy–Schwarz inequality respectively, gives

$$\begin{aligned} ||f - s_f||_\psi^2 = \langle f, f - s_f \rangle_\psi &= \sum_{k=m}^{\infty} \sum_{l=1}^{N_{k,d}} \frac{\hat{f}_{k,l} \cdot (\hat{f}_{k,l} - (\hat{s_f})_{k,l})}{\hat{c}_k} \\ &\leqslant \left(\sum_{k=m}^{\infty} \sum_{l=1}^{N_{k,d}} \frac{|\hat{f}_{k,l}|^2}{\hat{c}_k^2} \right)^{1/2} \left(\sum_{k=m}^{\infty} \sum_{l=1}^{N_{k,d}} (\hat{f}_{k,l} - (\hat{s_f})_{k,l})^2 \right)^{1/2} \\ &\leqslant ||f||_{\psi*\psi} \cdot ||f - s_f||_{L_2(S^{d-1})}. \qquad \square \end{aligned}$$

With this in place we can provide the following improved error bound.

Theorem 3.8. *Assume the same set up as in Theorem* 3.4 *and assume further that the target function f belongs to $H_{\psi*\psi}$. Then we have*

$$||f - s_f||_{L_p(S^{d-1})} \leqslant C^2 \cdot h^{\alpha + \frac{d-1}{2} + \frac{d-1}{p}} ||f||_{\psi*\psi}, \quad \text{for } p \in [2, \infty) \tag{3.15}$$

and

$$||f - s_f||_{L_p(S^{d-1})} \leqslant C^2 \cdot h^{\alpha + d - 1} ||f||_{\psi*\psi}, \quad \text{for } p \in [1, 2], \tag{3.16}$$

where C is the constant, independent of h, from Theorem 3.4.

Proof. Since $f \in H_{\psi*\psi} \subset H_\psi$ we have, from Theorem 3.4 with $p = 2$, that

$$||f - s_f||_{L_2(S^{d-1})} \leqslant C \cdot h^{\frac{\alpha}{2} + \frac{d-1}{2}} ||f - s_f||_\psi,$$

substituting this into (3.14) gives

$$||f - s_f||_\psi^2 \leqslant C h^{\frac{\alpha}{2} + \frac{d-1}{2}} ||f||_{\psi*\psi} ||f - s_f||_\psi,$$

cancelling the factor $||f - s_f||_\psi$ gives

$$||f - s_f||_\psi \leqslant C \cdot h^{\frac{\alpha}{2} + \frac{d-1}{2}} ||f||_{\psi*\psi}. \tag{3.17}$$

Substituting this inequality into the results of Theorem 3.4, namely (3.11) and (3.12), proves the theorem. $\square$

Note. Due to the norm equivalence of native spaces and Sobolev spaces, we can recast the error bound (3.17) as

$$||f - s_f||_{W_2^\beta(S^{d-1})} \leqslant \widetilde{C} \cdot h^\beta ||f||_{W_2^{2\beta}(S^{d-1})},$$

where $2\beta = \alpha + d - 1$ and where the constant $\widetilde{C}$ is independent of h. Thus, as a by-product, our analysis provides a useful convergence result for the case where the interpolation error is measured in an appropriate Sobolev space norm.

Corollary 3.9. *Assuming the same set up as in Theorem* 3.8, *we have*

$$\|f - s_f\|_{L_\infty(S^{d-1})} \leqslant C \cdot h^{\alpha + \frac{d-1}{2}} \|f\|_{\psi * \psi}, \tag{3.18}$$

where C is a positive constant independent of h.

Proof. Since $f \in H_{\psi*\psi} \subset H_\psi$, we can appeal to Theorem 2.12 to deduce that there exists a constant $\mathscr{C}$ independent of h such that

$$\|f - s_f\|_{L_\infty(S^{d-1})} \leqslant \mathscr{C} \cdot h^{\alpha/2} \cdot \|f - s_f\|_\psi.$$

The proof is completed by substituting (3.17) into the above. □

4. Conclusions

In [9], a numerical investigation into the performance of the ZBF method is presented. In particular, the numerical evidence strongly suggests that if $\psi \in CSPD_m(S^{d-1})$ has α-Fourier decay and $f \in H_{\psi*\psi}$, then the optimal L_p-error bound has the form

$$\|f - s_f\|_{L_p(S^{d-1})} \leqslant C \cdot h^{\alpha+d-1} \|f\|_{\psi*\psi}, \quad p \in [1, \infty], \tag{4.1}$$

for some constant C independent of h. Comparing this result with our theoretical error bounds, (3.15) and (3.16), we find that we have complete agreement in the case of $p \in [1, 2]$. However, for $p > 2$, there is gap between the theoretical bound and the numerically observed bound. Indeed, the authors believe that the task of bridging this gap that is, replacing the factor $\frac{d-1}{p}$ in (3.15) with $\frac{d-1}{2}$, is a challenging puzzle and one which deserves further investigation.

Remark. Subsequent to the completion of this work we have received the preprint [12] in which an analysis for RBF interpolation is presented. The analysis applies to domains which satisfy certain technical conditions. Examples of such domains include the usual open bounded subsets of $\mathbb{R}^d$ with the cone condition (see [4]) and also the unit sphere. The resulting error estimates for the unit sphere are comparable to those derived in this paper, however our techniques are different.

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